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THREE DIMENSIONAL ANALYSIS FOR FREE VIBRATION OF RECTANGULAR COMPOSITE LAMINATES WITH PIEZOELECTRIC LAYERS

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1. INTRODUCTION

An analysis for the three-dimensional solutions of piezoelectric laminated plates has been conducted by several researchers. For example, Ray and Rao [1] and Ray and Samanta [2] have studied the exact static analysis of piezoelectric plates under cylindrical bending and finite dimension piezoelectric plates. Free vibration exact solutions of infinite-length piezoelectric plates under cylindrical bending were studied by Heyliger and Brooks [3]. Recently, Batra and Liang [4] have studied the forced vibration of piezoelectric laminates using the three-dimensional elasticity theory, but the piezoelectric layers have been modelled as thin surface films in this paper, so it is not strictly speaking a three-dimensional solution.

The three-dimensional methods used in the papers mentioned above mostly follow the strategy of Pagano [5, 6]. This method has proved effective for static problems, but it is very complicated for the three-dimensional dynamic problems of piezoelectric laminated plates of finite-length. Laura [7] used some simple polynomial approximations and a variational approach to study the piezoelectric flexural plate hydrophone. In this paper, free vibration of a finite length rectangular piezoelectric composite laminates has been investigated based on three dimensional linear elasticity and piezoelectricity without any simplification. The laminates can be composed of an arbitrary number of elastic and piezoelectric layers of orthotropic materials. The solution of the derived governing differential equations is obtained using the power series expansion method. The results show that the method is more effective than the Pagano's method mentioned above. Different boundary conditions are studied to model the direct and inverse piezoelectric effects. The natural frequencies and the shapes of modal distribution for free vibration are investigated. The results obtained can be used not only to assess various approximate theories, but also enhance the understanding of the dynamic behavior of piezoelectric structures.

2. GOVERNING EQUATIONS

The configuration of the rectangular laminate with piezoelectric layers is shown in Figure 1. The piezoelectric layers can be bonded to the surface as well as embedded in the laminate. The material of each layer is assumed to be orthotropic. The linear constitutive equations coupling the elastoelectric field are used in this paper, that is

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \qquad D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k, \tag{1}$$

where, C_{ijkl}, e_{kij}, ϵ_{ik} are the elastic constants, the piezoelectric coefficients and the dielectric constants respectively. E_k are the components of the electric field and D_i the electric displacements.

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The displacement components u_i, where $u_1 = u$, $u_2 = v$, $u_3 = w$, are related to the strain through

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2}$$

The electric field components can be related to the electric potential Φ using the relation

$$E_i = -\partial\Phi/\partial x_i. \tag{3}$$

The dynamic equilibrium equations and the charge equilibrium equations for each layer are given by

$$\sigma_{ij,j} = \rho\ddot{u}_i, \qquad D_{i,i} = 0. \tag{4}$$

Substituting equations (1–3) into 4, the governing differential equations of the piezoelectric laminated plate are obtained as follows:

$$\begin{gathered}
C_{11}u_{,xx} + (C_{12} + C_{66})v_{,xy} + (C_{13} + C_{55})w_{,xz} + C_{66}u_{,yy} + C_{55}u_{,zz} + (e_{31} + e_{15})\Phi_{,xz} = \rho u_{,tt},\\
(C_{12} + C_{66})u_{,xy} + C_{66}v_{,xx} + C_{22}v_{,yy} + C_{44}v_{,zz} + (C_{23} + C_{44})w_{,yz} + (e_{32} + e_{24})\Phi_{,yz} = \rho v_{,tt},\\
(C_{13} + C_{55})u_{,xz} + (C_{23} + C_{44})v_{,yz} + C_{55}w_{,xx} + C_{44}w_{,yy}\\
+ C_{33}w_{,zz} + e_{15}\Phi_{,xx} + e_{24}\Phi_{,yy} + e_{33}\Phi_{,zz} = \rho w_{,tt},\\
(e_{15} + e_{31})u_{,xz} + (e_{24} + e_{32})v_{,yz} + e_{15}w_{,xx} + e_{24}w_{,yy} + e_{33}w_{,zz} - \epsilon_{11}\Phi_{,xx} - \epsilon_{22}\Phi_{,yy} - \epsilon_{33}\Phi_{,zz} = 0.
\end{gathered} \tag{5}$$

At each interface between layers, continuity conditions of displacement, traction, potential and electric displacement must be enforced. The continuity conditions between the ith and $(i+1)$th layer can be expressed as

$$\begin{gathered}
u^i = u^{i+1}, \qquad v^i = v^{i+1}, \qquad w^i = w^{i+1}, \qquad \sigma_z^i = \sigma_z^{i+1},\\
\tau_{yz}^i = \tau_{yz}^{i+1}, \qquad \tau_{xz}^i = \tau_{xz}^{i+1}, \qquad D_z^i = D_z^{i+1}, \qquad \Phi^i = \Phi^{i+1}.
\end{gathered} \tag{6}$$

Simply supported conditions are adopted in this paper,

$$\begin{gathered}
x = 0, a, \qquad v = w = 0, \qquad \sigma_x = 0, \qquad \Phi = 0,\\
y = 0, b, \qquad u = w = 0, \qquad \sigma_y = 0, \qquad \Phi = 0.
\end{gathered} \tag{7}$$

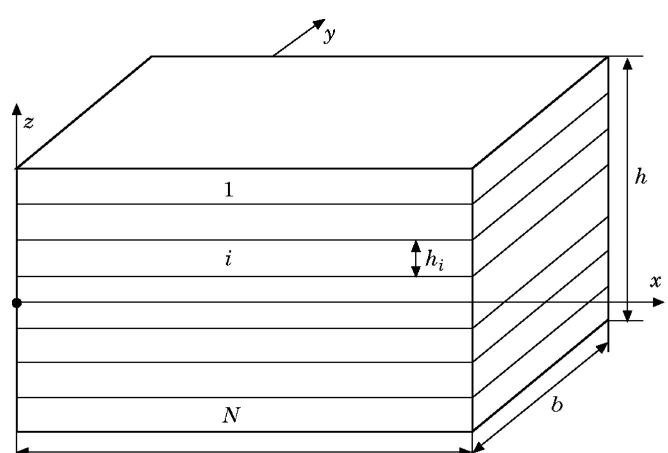

Figure 1. A N-layer rectangular composite laminate with piezoelectric layers.

TABLE 1

Natural frequencies of different mode of the laminate (1/s)

		$a = 20(h_1 + h_2 + h_3),\ b = 10a$	
m, n	Mode	Closed	Open
1, 1	I	800·5	804·8
	II	25788·0	25880·2
	III	64721·1	64715·0
1, 2	I	828·4	833·2
1, 3	I	874·4	880·1
2, 1	I	3107·4	3122·8
2, 2	I	3143·9	3159·9
2, 3	I	3185·9	3202·3
3, 1	I	6747·9	6777·1
3, 2	I	6808·0	6838·2
3, 3	I	6853·9	6884·6

For free vibration, the boundary conditions on the top and bottom surfaces are traction free, which can be stated as

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0. \tag{8}$$

In addition to the mechanical boundary condition, the electric surface conditions must be satisfied. When the direct piezoelectric effect is considered, the surface of piezoelectric layer is charged free. While the inverse piezoelectric effect is considered, the top and bottom surface of the piezoelectric layer are grounded at zero potential. These two cases are termed open and closed circuit.

$$D_z = 0 \text{ for open circuit}, \qquad \Phi = 0 \text{ for closed circuit} \tag{9}$$

3. EXACT SOLUTION

A set of power series solutions of displacement and electric potential satisfying the boundary conditions (7) is assumed to be

$$u = \sum_{k=0}^{\infty} A_x(k) z^k \cos px \sin qy\, e^{i\omega t}, \qquad v = \sum_{k=0}^{\infty} A_y(k) z^k \sin px \cos qy\, e^{i\omega t},$$

$$w = \sum_{k=0}^{\infty} A_z(k) z^k \sin px \sin qy\, e^{i\omega t}, \qquad \Phi = \sum_{k=0}^{\infty} A_\Phi(k) z^k \sin px \sin qy\, e^{i\omega t}, \tag{10}$$

where ω is the natural frequency, $p = m\pi/a$, $q = n\pi/b$, $m, n = 1, 2, \dots, \infty$.

After substituting equations (10) into the governing equations (5), the recurrence relations of $A_x(k)$, $A_y(k)$, $A_z(k)$, $A_\Phi(k)$ are obtained:

$$C_{55}(k+2)(k+1)A_x(k+2) = (C_{11}p^2 + C_{66}q^2 - \rho\omega^2)A_x(k) + (C_{12} + C_{66})pqA_y(k)$$
$$+ (C_{13} + C_{55})p(k+1)A_z(k+1) - (e_{31} + e_{15})p(k+1)A_\Phi(k+1),$$
$$C_{44}(k+2)(k+1)A_y(k+2) = (C_{66}p^2 + C_{22}q^2 - \rho\omega^2)A_y(k) + (C_{12} + C_{66})pqA_x(k)$$
$$+ (C_{23} + C_{44})q(k+1)A_z(k+1) - (e_{32} + e_{24})q(k+1)A_\Phi(k+1),$$

$$(e_{33}^2 + C_{33}\epsilon_{33})(k+2)(k+1)A_z(k+2) = [\epsilon_{33}(C_{13}+C_{55}) + e_{33}(e_{15}+e_{31})]p(k+1)A_x(k+1)$$
$$+ [\epsilon_{33}(C_{23}+C_{44}) + e_{33}(e_{24}+e_{32})]q(k+1)A_y(k+1)$$
$$+ [\epsilon_{33}(C_{55}p^2 + C_{44}q^2 - \rho\omega^2) + e_{33}(e_{15}p^2 + e_{24}q^2)]A_z(k)$$
$$+ [\epsilon_{33}(e_{15}p^2 + e_{24}q^2) - e_{33}(\epsilon_{11}p^2 + \epsilon_{22}q^2)]A_\Phi(k)$$

$$(e_{33}^2 + C_{33}\epsilon_{33})(k+2)(k+1)A_\Phi(k+2) = [\epsilon_{33}(C_{13}+C_{55}) - C_{33}(e_{15}+e_{31})]p(k+1)A_x(k+1)$$
$$+ [e_{33}(C_{23}+C_{44}) - C_{33}(e_{24}+e_{32})]q(k+1)A_y(k+1)$$
$$+ [e_{33}(C_{55}p^2 + C_{44}q^2 - \rho\omega^2) - C_{33}(e_{15}p^2 + e_{24}q^2)]A_z(k)$$
$$+ [e_{33}(e_{15}p^2 + e_{24}q^2) + C_{33}(\epsilon_{11}p^2 + \epsilon_{22}q^2)]A_\Phi(k). \quad (11)$$

There are eight unknowns for each piezoelectric layer i.e., $A_x(0)$, $A_x(1)$, $A_y(0)$, $A_y(1)$, $A_z(0)$, $A_z(1)$, $A_\Phi(0)$ and $A_\Phi(1)$, and six unknowns for each elastic layer i.e., $A_x(0)$, $A_x(1)$, $A_y(0)$, $A_y(1)$, $A_z(0)$, $A_z(1)$ because the dynamic governing equations are second order

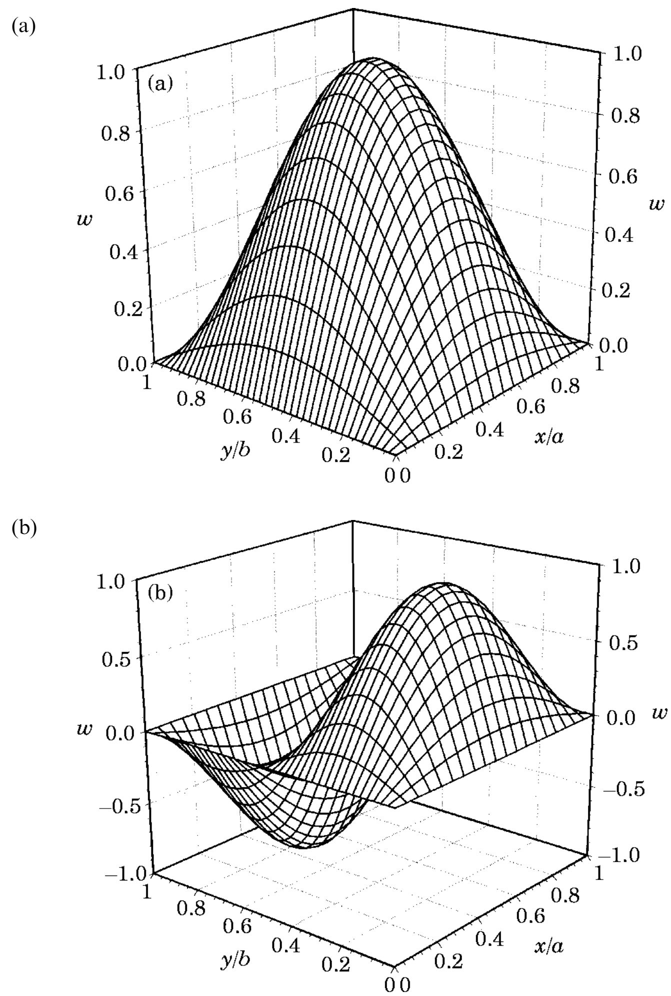

Figure 2. (a) and (b).

(c)

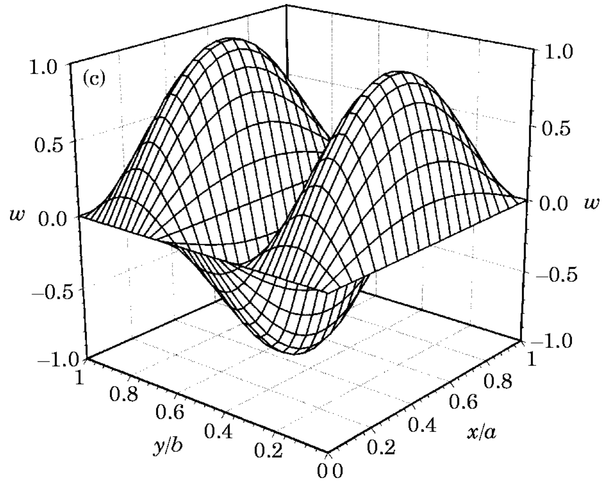

Figure 2. Distribution of the normalized displacement W of a piezoelectric laminate at the natural frequencies (a) ω_{11}, (b) ω_{12} and (c) ω_{13}.

partial differential equations. Therefore, the recurring relations of $A_x(k)$, $A_y(k)$, $A_z(k)$, $A_\Phi(k)$, can be expressed as the following forms in terms of the unknowns.

$$\begin{bmatrix} A_x(k) \\ A_y(k) \\ A_z(k) \\ A_\Phi(k) \end{bmatrix} = \begin{bmatrix} H_x(k,1) & H_x(k,2) & \cdots & H_x(k,8) \\ H_y(k,1) & H_y(k,2) & \cdots & H_y(k,8) \\ H_z(k,1) & H_z(k,2) & \cdots & H_z(k,8) \\ H_\Phi(k,1) & H_\Phi(k,2) & \cdots & H_\Phi(k,8) \end{bmatrix} \begin{bmatrix} A_x(0) \\ A_x(1) \\ \vdots \\ A_\Phi(0) \\ A_\Phi(1) \end{bmatrix} \tag{12}$$

where $H_x(0,1) = H_y(0,3) = H_z(0,5) = H_\Phi(0,7) = 1$ for $k = 0$ and $H_x(1,2) = H_y(1,4) = H_z(1,6) = H_\Phi(1,8) = 1$ for $k = 1$, the other terms in equation (12) are zero.

When $k > 1$, $H_x(k,l)$, $H_y(k,l)$, $H_z(k,l)$, $H_\Phi(k,l)$, $(k = 0, 1, \ldots, \infty, l = 1, 2, \ldots, 8)$ can be obtained by a set of recurring relations similar to equations (11). The recurring relations about H can be easily obtained by replacing A in equations (11) with H. Substituting equations (12) and (10) into equation (1), one gets

$$\sigma_z = \sum_{k=0}^{\infty} [-C_{13}pA_x(k) - C_{23}qA_y(k) + C_{33}(k+1)A_z(k+1) + e_{33}(k+1)A_\Phi(k+1)] \times z^k \mathrm{e}^{\mathrm{i}\omega t} \sin px \sin qy,$$

$$\tau_{yz} = \sum_{k=0}^{\infty} [C_{44}(k+1)A_y(k+1)) + C_{44}qA_z(k) + e_{24}qA_\Phi(k)] \times z^k \mathrm{e}^{\mathrm{i}\omega t} \sin px \cos qy,$$

$$\tau_{xz} = \sum_{k=0}^{\infty} [C_{55}(k+1)A_x(k+1) + C_{55}pA_z(k) + e_{15}pA_\Phi(k)] \times z^k \mathrm{e}^{\mathrm{i}\omega t} \cos px \sin qy,$$

$$D_z = \sum_{k=0}^{\infty} [-e_{31}pA_x(k) - e_{32}qA_y(k) + e_{33}(k+1)A_z(k+1) - \epsilon_{33}(k+1)A_\Phi(k+1)] \times z^k \mathrm{e}^{\mathrm{i}\omega t} \sin px \sin qy. \tag{13}$$

If the layer is not piezoelectric, the corresponding dynamic governing equations can be derived easily by assuming the piezoelectric coefficients e_{ij} to be zero. The details of the derivation and the relevant equations are not given here.

The eigenequation of free vibration can be obtained by satisfying the boundary conditions on the outer surface and the continuity conditions on the interface between two adjacent layers. If the laminate has m piezoelectric layers and n purely elastic layers, there are $(8m + 6n)$ total unknowns which can be expressed in matrix form as

$$[\mathbf{B}]\{\mathbf{A}\} = 0. \tag{14}$$

Here, the $\{\mathbf{A}\}$ represents the unknowns ($A_x(0)$, $A_x(1)$, $A_y(0)$, $A_y(1)\ldots$), the matrix[**B**] is the function of the natural frequency ω. If equations (14) have non-trivial solutions, the matrix [**B**] must equal zero. That is

$$\text{Det}\,[\mathbf{B}] = 0. \tag{15}$$

The natural frequency can be determined by equation (15).

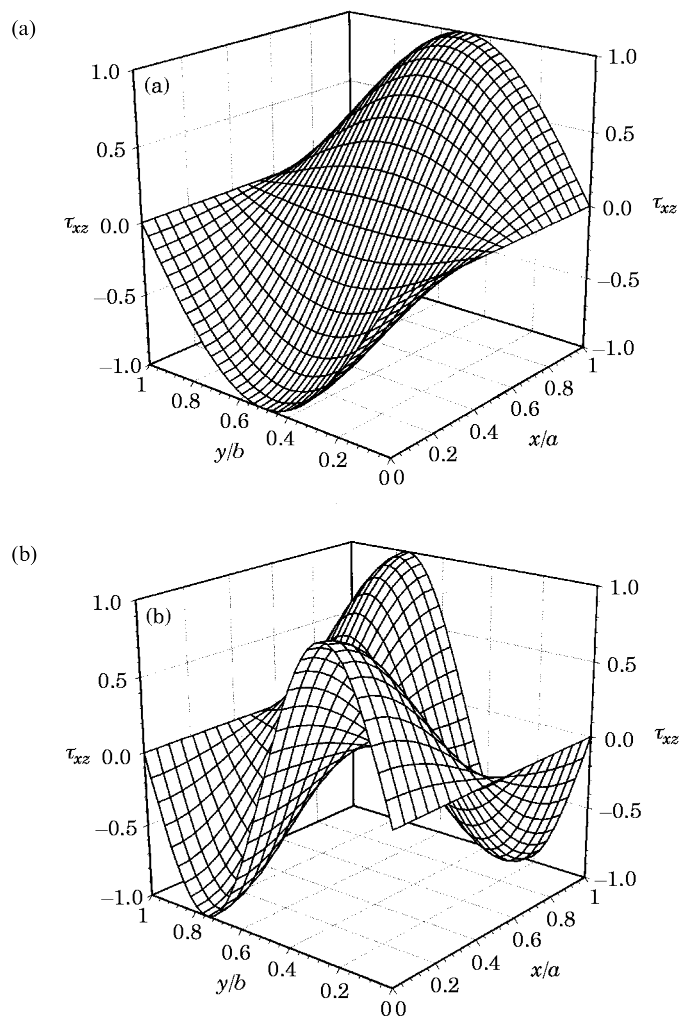

Figure 3. (a) and (b).

(c)

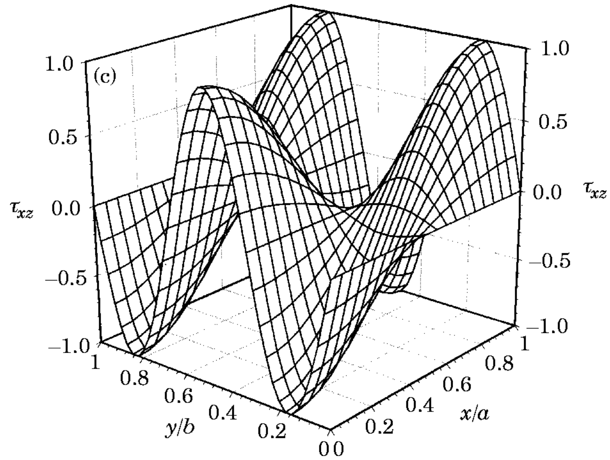

Figure 3. Distribution of the normalized stress τ_{xz} of a piezoelectric plate at the natural frequencies (a) ω_{11}, (b) ω_{12} and (c) ω_{13}.

4. NUMERICAL EXAMPLES

Example

The topmost layer is the piezoelectric layer. The piezoelectric material used here comes from the Heyliger and Brooks [3] paper

$$[\mathbf{C}] = \begin{bmatrix} 81{\cdot}3 & 0{\cdot}329 & 0{\cdot}432 & 0 & 0 & 0 \\ 0{\cdot}329 & 81{\cdot}3 & 0{\cdot}432 & 0 & 0 & 0 \\ 0{\cdot}432 & 0{\cdot}432 & 64{\cdot}5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25{\cdot}6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25{\cdot}6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30{\cdot}6 \end{bmatrix} \times 10^9 (\mathrm{pa}),$$

$$e_{31} = e_{32} = -5{\cdot}20\,\frac{\mathrm{c}}{\mathrm{m}^2},$$

$$e_{33} = 15{\cdot}08\,\frac{\mathrm{c}}{\mathrm{m}^2}, \qquad e_{15} = e_{24} = 12{\cdot}72\,\frac{\mathrm{c}}{\mathrm{m}^2}, \qquad \epsilon_{11} = \epsilon_{22} = 1{\cdot}475\epsilon_0,$$

$$\epsilon_{33} = 1300\epsilon_0, \qquad \epsilon_0 = 8{\cdot}854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}, \qquad \rho = 7{\cdot}5 \times 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

The second and third layers are the composite matrix and the layout is (0/90); the material properties are found in Pagano [5]. The thicknesses of the three layers h_1, h_2 and h_3 ($h_1 = h_2 = h_3 = 0{\cdot}01$ m) in this example and $E_1 = 25E_0$, $E_2 = E_3 = E_0$, $G_{12} = G_{13} = 0{\cdot}5E_0$, $G_{23} = 0{\cdot}2E_0$, $v_{12} = v_{13} = v_{23} = 0{\cdot}25$, $E_0 = 6{\cdot}84976 \times 10^9$ Pa.

The different modal frequencies of the laminate are shown in Table 1. The results indicate that the frequencies generated under the closed circuit condition are lower than those under the open circuit condition. That means the influence of the direct piezoelectric effect and the inverse piezoelectric effect on natural frequencies is different.

The distributions of the non-dimensional displacement W and shear stress τ_{xz} at the interface between the piezoelectric and the elastic layers of the laminate at the natural frequencies ω_{11}, ω_{12}, ω_{13} are plotted in Figures 2(a), (b), (c) and Figures 3(a), (b), (c), respectively. The distributions of electric potential Φ at the piezoelectric layer surface and shear stress τ_{yz} at the interface are similar to the figures.

It can be seen that the shear stress τ_{xz}, τ_{yz} at points on the interface between the piezoelectric and elastic layers reaches the maximum in the region near the edges. This is similar to the results for beams computed by Crawly and Deluis [8] and by Wang and Rogers [9] who used the classical laminate plate theory, but it is more directly perceived here.

5. CONCLUSION

An exact analysis for the free vibration and forced vibration of a finite-length rectangular orthotropic piezoelectric laminate based on three dimensional elasticity and piezoelectricity is presented. The solution is obtained by the power series expansion method. The method has some merits, such as simplicity, convenience and better convergence for computation. Different boundary conditions are studied to model the direct and inverse piezoelectric effects. The natural frequencies and the modal distributions for free vibration of a three-ply orthotropic piezoelectric laminate are investigated. The results show that the influence of the direct and inverse piezoelectric effects on natural frequencies is different.

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